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About Lorentz invariance in a discrete quantum setting

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Abstract: A common misconception is that Lorentz invariance is inconsistent with a discrete spacetime structure and a minimal length: under Lorentz contraction, a Planck length ruler would be seen as smaller by a boosted observer. We argue that in the context of quantum gravity, the distance between two points becomes an operator and show through a toy model, inspired by loop quantum gravity, that the notion of a quantum of geometry and of discrete spectra of geometric operators, is not inconsistent with Lorentz invariance. The main feature of the model is that a state of definite length for a given observer turns into a superposition of eigenstates of the length operator when seen by a boosted observer. More generally, we discuss the issue of actually measuring distances taking into account the limitations imposed by quantum gravity considerations and we analyze the notion of distance and the phenomenon of Lorentz contraction in the framework of "deformed (or doubly) special relativity" (DSR), which tentatively provides an effective description of quantum gravity around a flat background. In order to do this we study the Hilbert space structure of DSR, and study various quantum geometric operators acting on it and analyze their spectral properties. We also discuss the notion of spacetime point in DSR in terms of coherent states. We show how the way Lorentz invariance is preserved in this context is analogous to that in the toy model.

Keywords: Space-Time Symmetries, Models of Quantum Gravity, Non-Commutative Geometry.

"It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time"
Snyder, 1947

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Contents

1. Introduction

Gravitation is geometry and measurements of distances are measurements of properties of the gravitational field. Geometric quantities are the observable properties of the gravitational field. If the gravitational field is quantized, then this would result in geometric observables being quantum operators of which it is sensible to compute the spectrum and to find the set of eigenstates. A first expected effect of the combination of the principles of General Relativity and Quantum Mechanics is the appearance of a fundamental length scale, the Planck length $l_p = \sqrt{\hbar G/c^3}$, and of a fundamental time scale, the Planck time $\tau_p = \sqrt{\hbar G/c^5}$. These are supposed to represent a lower bound on the measurement of spacelike and timelike distances in a quantum gravity setting, where both the quantum and dynamical properties of the geometry are taken into account, and the very notion of distance is expected to lose physical meaning below such scales. Also, these new fundamental constants would act as a natural cutoff for any field living on a manifold endowed with a quantum geometry [2], also explaining puzzling features of black hole physics [3].

Provisional results confirming these expectations come from both quantum gravity, in the canonical [4, 5] as well as in some versions of the covariant approach [6, 7], and string theory [8].

Moreover, it is a intriguing possibility that, as it happens for many classically continuous observables, geometric operators would turn out to have a discrete spectrum in a fully quantized theory. Geometric operators such as lengths or areas, or time intervals, would then assume, when measured on states of the quantum gravitational field, only a discrete set of possible values, with probabilities computable within the theory itself. This possibility, again, is realized in both loop quantum gravity and spin foam models.

This, of course, has to be true also if the gravitational field is in a particular quantum state being a "Minkowski quantum state", in the sense that it approximates a flat Minkowski geometry on scales larger than the Planck scale, i.e. in a semiclassical/continuum approximation, provided we do not neglect completely the quantum nature of geometric observables, by taking $\hbar \rightarrow 0$.

Needless to say, this is a very different picture of spacetime geometry with respect to the classical one we are accustomed with, particularly with respect to the simple picture of spacetime geometry in a classical flat Minkowski background.

This conceptual jump naturally raises confusion and a host of questions.

In particular, what is the relationship between this lowest bound in distances, or the possible discreteness of the spectrum of lengths, and the usual continuous symmetries of classical GR or special relativity. We know that, in a flat classical geometry, a spacelike length at rest with respect to a given observer will result in being contracted when measured by a second observer related to the first by a Lorentz transformation (boost), while time intervals are dilated by the same kind of transformation. The contraction and dilatation are moreover governed by a continuous parameter, the boost rapidity. It seems then that both the idea of a minimal length and that of a discrete set of possible values for lengths and time intervals are at odds with what we know about Lorentz symmetry.

If an observer has verified the existence of a minimal length being the Planck one for a given object, a second boosted observer will see a smaller length for the same object, as a result of Lorentz contraction. If the first observer has verified the existence of a discrete spectrum for the length of that object, the second boosted observer will fail to detect this discreteness because of the continuous nature of the Lorentz symmetry. In any case, it seems that neither the lowest planckian bound for distances, nor their spectrum can possibly be an invariant property of the theory, if we require it to be Lorentz symmetric in the flat regime.

Of course none of these observations is conclusive, and the problem as explained above is pretty naively posed, since it pretends to apply what we know about Lorentz symmetry in flat Minkowski spacetime to a fully quantum geometric theory in which a breakdown of all conventional notions of smooth geometry are likely to take place.

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Nevertheless, any quantum gravity model predicting either discrete spectra of geometric operators or minimal and non-zero eigenvalues for them (or both) should face these issues and show how the apparent paradox is solved.

Does a quantum gravity theory with an invariant length and a discrete spectrum for geometric observables necessarily break Lorentz symmetry or necessarily require some sort of modification/deformation of it? The answer, as we will see, is simply "no".

In this paper, we tackle this issue of the action of length contraction/dilatation in a quantum gravity setting in a simplified toy model inspired by loop quantum gravity, and then in the context of deformed (or doubly) special relativity, considered as an effective description of some features of a full quantum gravitational theory.

1.1 Loop quantum gravity discreteness

Loop quantum gravity [4, 5] is a well-developed theory for dealing with the quantum properties of geometry, and as such all these issues are to be addressed in it. Indeed, in the loop quantum gravity context we see at work both the features we anticipated above: it predicts a precise spectrum for geometrical quantities, such as lengths, areas and volumes, and in many cases these spectra turn out to be discrete, and result in having a lowest bound at the Planck scale. Here we focus on what happens to lengths, and therefore we analyze the situation in $2+1$ gravity, where lengths operators are most easily analyzed when the theory is quantized in the loop approach. An additional reason for dealing with $2+1$ gravity is that the theory (in the absence of a cosmological constant) is flat, and so the implementation of Lorentz symmetry is most easily analyzed. Of course, we face the same conceptual/technical issues as outlined above.

In particular, in the lorentzian version of $2+1$ loop quantum gravity we have a prediction of a continuous spectrum for spacelike distances and of a discrete spectrum for timelike intervals [9]. This is also fully consistent with the picture obtained from the corresponding covariant approach, i.e. from lorentzian spin foam models in 3d [10]. Indeed, in this approach the length operator acts diagonally on spin networks whose edges are labeled by representations of $\mathrm{SO}(2,1)$. one edge As there are two types of (principal) representations of $\mathrm{SO}(2,1)$, the spectrum of the length operator is

$$\hat{L}\,\Psi = l_p\sqrt{\rho^2+\frac{1}{4}}\,\Psi \qquad \text{and} \qquad \hat{L}\,\Psi = i\tau_p\,\epsilon\,\sqrt{-n(n-1)}\,\Psi\,, \tag{1.1}$$

where $\rho\in\mathbb{R}_+$ or $(\epsilon=\pm1, n\in\mathbb{N})$ label the (unitary) representations of $\mathrm{SO}(2,1)$.

Let us point out that there exists a quantization ambiguity resulting from the regularization procedure. This leads to an alternative length spectrum:

$$\hat{L}_s\,\Psi = l_p\,\rho\,\Psi \qquad \text{and} \qquad \hat{L}\,\Psi = \tau_p\,\epsilon\,i\left(n-\frac{1}{2}\right)\Psi\,. \tag{1.2}$$

The length spectrum with continuous representations used has the interpretation of a quantization of spacelike distances, it is continuous and, at least in the first case, presents a lowest bound on allowed values corresponding to (half) the Planck length. When discrete representations are used we have a quantization of timelike distances (intervals), the spectrum is discrete and, at least in the second case, it presents a lowest bound on the allowed values, equal again to (half) the Planck time[1].

Of course, the above spectra are for the length of spacetime intervals in flat space (because 3d gravity admits only flat solutions), and these are not supposed to change under boosts, which are in fact isometries of the flat metric. Therefore obtaining a discrete spectrum like the one given above is not at odds with Lorentz invariance. However it suggests that other geometric quantities, that are not invariant under boosts, and indeed depend on a choice of observer, like time intervals or space distances, may have a discrete spectrum as well, and the question is whether such quantities maintain their discrete features or not under boosts. The analogy is with the usual coordinate representation of the spacetime infinitesimal interval in Minkowski space: $ds^2 = -dt^2 + dx^2 + dy^2$, in a given coordinate frame, so for a given observer; its integral between two events A and B gives their invariant spacetime distance (square) L^2_{AB} that is the result of a time interval T^2_{AB} and a space distance S^2_{AB}, as $L^2_{AB} = -T^2_{AB} + S^2_{AB}$; of course, neither of the last two quantities is observer independent and indeed it is these two that change under boosts, i.e. in going to another observer (coordinate choice), but leaving the full spacetime length invariant. But if L^2_{AB} is discrete, then it is likely that T^2_{AB} and S^2_{AB} can be discrete as well; the question is what is their behavior under boosts. This analogy will be exploited in constructing a simplified model of an operator representing time interval measurements in the following.

We are then faced with the following theoretical/mathematical issue: is it possible to have a consistent length contraction or time dilatation for boosted observers in the presence of the above (discrete) length spectrum? Assuming that an inertial observer measures a quantum length with results governed by a spectrum of the kind given above, what would a boosted observer measure for the same object? In other words, how does a Lorentz transformation change the previous equation?

There are two possibilities. It is possible that the spectrum will change, meaning that it does not represent a Lorentz invariant property of the theory; this may happen if to a boosted observer it corresponds a change of the representation of the operators defining the quantum theory, giving an inequivalent quantization; in the context of the SU(2)-based loop quantum gravity this would be for example a change of the Immirzi parameter. The second possibility is to have an invariant spectrum on which both the initial and the boosted observer agree, although they may disagree on the expression for both the length operator

[1]These are the formal results of the theory regarding the spectrum of distances, and refer to observables which can be in principle measured by suitably planned (and precise enough) experiments. However, much care should be used in interpreting them, and in relating them to actual observations. The question is: what is really measurable (experimentally)? The issue is what a correct operational definition of a spacelike or timelike distance is, what would be a concrete and realistic procedure for measuring them. Consider for example that in special relativistic settings the measurement of spacelike distances is achieved through emission and subsequent reception of light signals and measurement of timelike intervals giving information about the traveled distance by the light. If anything similar is true also in a quantum gravity context, then a discreteness of measured timelike intervals may imply a discreteness of measured spacelike distances, when an appropriate operational definition of measurement is given.

and the quantum state of the gravitational field. In this situation:

$$\hat{L}_{\text{boosted}}(\beta) = U(\beta)\,\hat{L}\,U^{-1}(\beta) \qquad \text{and} \qquad \Psi_{\text{boosted}}(\beta) = U(\beta)\,\Psi\,, \tag{1.3}$$

where $U(\beta)$ is the operator corresponding to a Lorentz boost in an appropriate representation of the Lorentz group, but still

$$\hat{L}_{\text{boosted}}\,\Psi_{\text{boosted}} = L\,\Psi_{\text{boosted}}\,. \tag{1.4}$$

The usual Lorentz contraction/dilatation will be found for the mean value of the operator in any given state. Of course this is possible only if U is in a unitary representation of the Lorentz group (remember that the only representations labeling the edges of spin networks in loop quantum gravity are the unitary ones) and if the boosted and un-boosted length operators do not commute, $[\hat{L}, \hat{L}_{\text{boosted}}] \neq 0$. This possibility was indeed proposed, as a way to solve the apparent paradox, and investigated in [11].

This second resolution of the naive paradox, leading to an Lorentz invariant spectrum of quantum geometric operators, is not at all a result of pure wishful thinking or a lucky but unlikely to happen possibility. It is nothing else than what happens in ordinary quantum mechanics as a natural resolution of an analogous apparent paradox, in which the theory predicts a discrete spectrum with lowest bound for an observable that is transformed by a continuous transformation when different observers are considered: angular momentum. The angular momentum of a particle may be quantized and its spectrum is discrete, with its lowest possible eigenvalue being 1/2 for a fermionic particle. Rotations are continuous transformations and are represented in the quantum theory by quantum operators labeled by a continuous angular parameter θ. However, the spectrum of the angular momentum is nevertheless a rotational invariant property of the theory, in the sense that two observers related by a rotation one to the other will measure a different operator on a different state of the particle, but the set of allowed eigenvalues obtained will be the same. Let us apply the same reasoning to our situation and let us consider a ruler in the proper state of length l_P for a given observer, so this observer defines the rest frame for the ruler. Now, a boosted observer will see a superposition. This means that when measuring the length of the ruler, she might measure a zero distance or l_P or even $2l_P$ and that only the average value of his/her measurements will be l_P/γ and obey the classical contraction rule. What does it mean that the observer can see a zero distance? A priori, one would say that if two objects/particles/excitations are distinct in space, then they will be whatever the observer. In fact, one needs to remember that the theory we are describing right now is an effective theory, in a semi-classical context, with classical objects but quantum (flat) spacetime. In a fully quantum context, one expect objects to be described by wave packets, and is the two wave packets are overlapping too much, then it is plausible that an observer could not distinguish them.

In this paper we present a toy model meant to mimic closely, but in a simplified way, the measurement of time intervals in loop quantum gravity in 3 spacetime dimensions. We then introduce boost operators which reproduce the usual time dilatation on the average, but such that however, as $[\widehat{L}, \widehat{L}_{\text{boosted}}] \neq 0$, an eigenstate for an observer will become a superposition for any boosted observer. In other words, we give an explicit realization of the second possibility envisaged above, although in a simpler context than the full theory, of which however we reproduce the relevant features. Let us emphasis that information about the length is not lost during the process of going from an observer to the other: a pure state stays a pure state and does not become mixed.

Our system may be thought of as a model for a a quantum clock (timelike rod) living in a flat quantum geometric spacetime in which pointlike test (because they do not affect the quantum geometry of spacetime) particles have been introduced to give a physical meaning to the time intervals considered [12].

Then we map this model to a model about discrete spacelike distances and recover the usual length contraction. Interestingly, this involves mapping SU(1, 1) to SU(2), following a proposal by 't Hooft [13].

1.2 Deformed special relativity and a Lorentz invariant minimal length

A second context in which it is interesting to try to address the issues raised in the beginning is that of deformed special relativity theories (DSR) [14, 15]. DSR was especially introduced to address the second of these issues: have a fixed invariant minimal length which can be measured by the same experiment by any observer, and on the value of which any observer would agree. The context is that of flat Minkowski space and this is realized by means of a quantum deformation of the Poincaré algebra of symmetries [16].

DSR can be considered [17, 18] an effective flat regime of the full quantum gravity theory, when $G \to 0$ but l_p stays fixed, therefore $\hbar$ is sent to infinity. We are then neglecting both the curvature of spacetime and the dynamical aspects of the gravitational interaction, thus of the geometry, while retaining its quantum features; it is then interpretable as a model for the kinematics of any quantum gravity theory. It has been indeed shown, both in the canonical framework [19] and in the spinfoam context [20], to encode the algebra of observables associated to a particle in $2+1$ quantum gravity, that, we recall, is flat.

Therefore we want to consider the notion of Lorentz contraction/dilatation also in the DSR framework, on the one hand to see in which way the naive paradox associated to the existence of a minimal length/time interval is solved, and on the other hand to compare it with the LQG framework.

But what does DSR say about length contraction/dilatation?

There are many issues to understand, before being able to answer this question in full. A first problem is in the operational definition of what a measurement of distances is in this context. Indeed, the speed of light is found to vary (with the energy of the photon), so how can we measure a (spacelike) distance? However, the most basic problem from our point of view is that it is not clear what a length is in a DSR theory. The operators encoding the position of a point become non-commutative, under the assumed deformation of the Poincaré algebra of symmetries and of the momentum space, obeying the commutation relations of the κ-deformed Minkowski algebra. Therefore we simply cannot localize a point in spacetime, so that the trivial definition of a spacetime distance as the length of a curve connecting two points is not available. To tackle the problem, we see two approaches. On the one hand we make use of the geometric picture of DSR theories based on a curved

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space of momenta, and on the associated group of symmetries. We write a (spacetime) length operator as acting on the momentum space and study its spectrum. Our working definition of distance in a given direction will then be given by the very time or space coordinate of a point on the non-commutative spacetime, and we study their spectrum and transformation properties. On the other hand, we introduce spacetime coherent states corresponding to a point being localized around a spacetime point $(x^0, \vec{x})$, with minimal uncertainty. Of course, in this way the origin would assume a privileged status, as it is in all of DSR, representing the point of view of some observer in its rest frame. The study of the transformation properties of these states under boosts, which is left for future work, would represent another approach to the study of Lorentz contraction in DSR. Finally, we compare the results obtained in this context with those of the toy (loop inspired) model. It will be clear that the two approaches agree perfectly regarding the general way a quantum discrete geometry can still be Lorentz invariant, although the details are of course different.

2. About length spectra in loop quantum gravity

2.1 Measuring distances and length contraction

The most basic experiment to measure a distance between two points is to do the "radar" experiment. First, one must assume that the two points/particles/objects, A and B, are static with respect to one each other, so that they can effectively be considered as a ruler. Then the observer send a ray of light from A to B, which gets reflected back to A. Finally, the distance AB is the time T of flight of the photon:

$$d_{AB} = \frac{T}{2} c\,.$$

Now the precision of the measurement is determined by two parameters: the wavelength of the photon λ and the frequency ω of the clock. On the one hand, $\delta d \geq \lambda$ and on the other hand, $\delta d \geq c/\omega$. To increase the precision of the measurement, therefore, one needs to increase the energy of both the photon and the oscillator which we use as the clock.

Now there are strong reasons, coming from taking into account the basic properties of gravity and the features of quantum mechanics, to believe that one can not increase the energy localized in a given small region of spacetime beyond the Planck scale: neither the wavelength of the photon can be smaller that the Planck length, nor its energy can be greater than the Planck energy, so that $\delta T \geq t_P$ and $\delta d \geq l_P$.

From this measurement procedure, we can deduce a couple of facts. First, spacelike lengths and space distances are not observable as such: we effectively measure time intervals, i.e. particular timelike lengths, so that time discreteness will imply the discreteness of measured distances. Moreover, using a clock to measure time intervals naturally implies that our result will actually be discrete. And if we want to make more precise measurements, we are stuck with the experimental resolution bound given by the Planck time. One of the things that loop quantum gravity achieves is to encode this bound directly in the spectrum of time intervals, so among the very properties of the quantum gravitational

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field: it is a result of the theory that time intervals are quantized as $T = n \times t_P$ with t_P as the minimal time interval. More precisely, it is not that it is hard to probe time intervals smaller than t_P: now, there doesn't exist a time interval shorter than t_P.

One can refine the measurement argument by taking into account the mass of the mirrors at the end of the ruler. Then one finds that the uncertainty (or precision) δd actually depends on the distance d that one measures. The precise relation in 3+1d reads:

$$\delta d \geq l_p \times \left(\frac{d}{l_P}\right)^{1/3} .$$

If the holographic principle holds, a similar uncertainty bound is ture but with a different exponent on the right end side,[2] coming from the fact that now the number of degrees of freedom scales with the area and not with the volume, as shown in [21]. In general, such refined relations read:

$$\delta d \geq l_P \times \left(\frac{d}{l_P}\right)^{\alpha} , \tag{2.1}$$

with $\alpha = 1/n$ in (n+1)-d ($\alpha = 1/3$ in 4d and $\alpha = 1/2$ in 3d). It should be interesting to check if we are able to recover such uncertainty bound in the context of quantum gravity theories. As we will see in the DSR section below, the uncertainty of the measurement of a distance actually depends on the considered distance, and we discuss how to possibly derive the previous formulae without yet being able to derive them rigourously.

Now, the usual length contraction/dilatation phenomenon is that different observers will see different lengths d_{AB} and time intervals T_{AB} depending on their speed with respect to the ruler. More precisely, if d_0 is the distance seen by the static observer (with respect to the ruler), then the length measured by an observer with speed v (in c unit) is $d = d_0/\gamma$ with $\gamma = 1/\sqrt{1-v^2} \geq 1$. However, as all measurements are after all time interval measurement, it might be more interesting to consider how these differ depending on the observer. The relevant phenomenon is then time dilatation, i.e. the fact that an observer moving with velocity v (again in c units) with respect to one which has measured a time interval T_0 between two events will measure for the same two events a time interval $T' = T_0\gamma \geq T_0$. This also entails that relativistic clocks will tick differently for different observers looking at them, with different frequency that is, with the moving observer seeing a slower clock (with frequency $1/T = 1/T_0\gamma$) than the observer at rest with respect to the clock itself (who sees a frequency $1/T_0$).

Also in this case, however, an operational definition of space and time intervals measurements, and a proper derivation of Lorentz contraction and dilatation involve considering light signals sent back and forth between two observers moving with different velocities in spacetime and not purely geometric considerations. In a quantum gravity context, obviously, such a careful derivation would be even more difficult to achieve. We can therefore

[2]Roughly the argument goes as follows. Let us be in n+1 spacetime dimension and try to evaluate the number of degrees of freedom in a volume of length d. Then the precision of the measurement δd on the distance d defines our concept of point i.e. of distinguishable excitations i.e. of fundamental degree of freedom. The number of degrees of freedom will therefore be $(d/\delta d)^n$, but should be equal to $(d/l_P)^{(n-1)}$ if the holographic principle is realized in nature

make us of the fact that the purely geometric analysis of how time and space intervals in the manifold are acted upon by the Lorentz group of isometries of the metric gives the same results of the physical analysis and confine ourselves to it. Of course we can take this easier path only in the classical setting or in simplified models of the quantum gravity regime, of which we have no real control; hopefully, the toy model we are going to present for time measurements and the effective framework of DSR are such that we can safely limit the discussion to geometric considerations only.

2.2 A Lorentz invariant discrete time spectrum

Let us start by writing a simple model for a quantum geometric clock in flat space, which reproduces the relevant features of 3d loop quantum gravity, and which gives a discrete time interval spectrum as a results of its readings; we would like to show that this spectrum may well be invariant under boosts, in spite of its discreteness, and that the the usual behavior under boosts (time dilatation) is recovered when appropriate in a quantum setting, i.e. for the mean values.

In slightly more details, we would like an operator $\widehat{T}$, measuring time intervals in Planck time units, with eigenvectors $|0\rangle, |1\rangle, |2\rangle, |3\rangle \ldots$, and boost operators B_γ which can take eigenvectors to generic superpositions but such that the average value of the boosted time interval obey (exactly) the law of time dilatation. We work in $2+1$ dimensions to make things simpler, but a corresponding analysis could be made for the 4-dimensional case.

Consider first, as an analogy, the expression for the spacetime length in ordinary special relativity in a canonical basis of coordinates, i.e. minkowskian cartesian coordinates: $C^2 = -T^2 + X^2 + Y^2$. What we are interested in is the equivalent of the time interval operator $\widehat{T}$, because this is what changes under boosts.

Now we have seen that in 3d loop quantum gravity, i.e. in a quantum geometry in 3d flat space, the spacetime length operator (square) is given by the Casimir of the 3d Lorentz group SU$(1,1)$: $C(\mathrm{SU}(1,1)) = -J_t^2 + J_x^2 + J_y^2$, where the J_i's are the canonical generators of the Lie algebra. The spectrum of the above is discrete or continuous depending on the representation one works in, and, as we have said, unitary discrete representations have then the interpretation of timelike distances and the unitary continuous one of spacelike distances.

We are interested in the behavior of time intervals under boosts for a given spacetime length, so let us fix a unitary representation of SU$(1,1)$; in this representation, of course, boost operators and their actions are represented unitarily.

The most natural choice for the operator $\widehat{T}$, looking at the above expression for the spacetime length operator, and keeping in mind the analogy with the above expression for the spacetime length in special relativity, is the generator J_t of the rotation of SU$(1,1)$. If one wants to have one, an intuitive picture for this choice is to think of this operator as measuring the spin of a spinning particle at rest, so moving along the time axis in Minkowski spacetime, a very fundamental model for a quantum clock indeed.[3]

[3]Of course, a more carefully defined modeling of a fundamental quantum clock of this type should assign

Now, we choose a unitary representation of SU$(1,1)$. A convenient basis for our purpose is the basis diagonalizing the operator J_t. More precisely, we represent SU$(1,1)$ on the space of C^∞ functions on the circle (which play then the role of the Hilbert space for this quantum system - the clock). The basis is $|n\rangle = e^{in\theta}$, where $\theta \in [0, 2\pi]$ is the coordinate on the circle and $n \in \mathbb{Z}$ is an integer (in order for the functions to be single-valued). The time interval operator J_t acts as $-i\partial/\partial\theta$ and has a discrete spectrum: $J_t|n\rangle = n|n\rangle$. Then $J_\pm = J_x \pm iJ_y$ will acts the left or right translations, acting as multiplications by $e^{\pm i\theta}$.

The exact action of the generators is:

$$\begin{cases} J_t|n\rangle = n\,|n\rangle\,, \\ J_+|n\rangle = \sqrt{\Omega + n(n+1)}\,|n+1\rangle\,, \\ J_-|n\rangle = \sqrt{\Omega + n(n-1)}\,|n-1\rangle\,, \end{cases} \tag{2.2}$$

where Ω is actually the eigenvalue of the Casimir operator $-J_t^2 + J_x^2 + J_y^2$ and labels the different representations. The (principal) unitary (irreducible) representations are of three kinds:

- the continuous series $\Omega = s^2 + 1/4 > 0, s \in \mathbb{R}_+$: the representation is spanned by all vectors $n \in \mathbb{Z}$.
- the positive discrete series $\Omega = -j(j-1), j \in \mathbb{N}^*$: it is a lowest weight representation spanned by vectors $n \geq j$.
- the negative discrete series $\Omega = -j(j-1), j \in \mathbb{N}^*$: it is a highest weight representation spanned by vectors $n \leq -j$.

In our toy model, we will use a j^+ representation. The spectrum of the operator $\widehat{T}$ will be discrete and bounded from below: there will be a minimal time interval of length j. Now let us consider the action of boosts on this operator and on the corresponding results of measurements. The action of boost operator J_y, say, on the time interval operator is given simply by

$$\widetilde{J}_t = e^{+i\frac{\eta}{2}J_y} J_t e^{-i\frac{\eta}{2}J_y} = \cosh\eta J_t + \sinh\eta J_x\,, \tag{2.3}$$

while the state (of the clock) as seen by a boosted observer is $|n\rangle \to e^{i\frac{\eta}{2}J_y}|n\rangle$.

First the operators J_t and $\widetilde{J}_t$ do not commute:

$$[J_t, \widetilde{J}_t] = [J_t, \sinh\eta J_x] \neq 0\,, \tag{2.4}$$

so that an eigenvector of J_t will not be an eigenvector of the boosted time interval operator $\widetilde{J}_t$: the eigenstates of the time interval operator in one frame are not invariant under boosts and are mapped to superpositions of eigenstates of the same operator. In other words, a state of definite time interval size for a given observer will be a superposition of such states for a boosted observer, for whom the time interval will then not have a definite value. What

to it a dynamics which is such that the system evolves monotonically through eigenstates of increasing eigenvalue [12]. However, the dynamics of the clock and a precise modeling of it is not relevant for the issue we want to address in this paper.

is *invariant* under boosts is the *spectrum* of the operator $\widehat{T}$ measuring timelike spacetime intervals. The continuity of the Lorentz transformation affects the way wave function distributions transform, but not the spectrum of the geometric quantities. We can also check that the mean value of the time interval operator simply transforms as expected:

$$\langle n|\widetilde{J}_t|n\rangle = n\cosh\eta = \cosh\eta\langle n|J_t|n\rangle\,. \tag{2.5}$$

As $\cosh\eta = \gamma$ for a boost, one can conclude that we recover the usual time dilatation law on the expectation values, and that it does not depend on the chosen $\mathrm{SU}(1,1)$ representation. We can go further and compute the variance of the length state seen by the two observers. First, obviously, we have $(\Delta J_t)^2 = 0$. Then:

$$\left(\Delta\widetilde{J}_t\right)^2 = \langle n|\widetilde{J}_t^{\,2}|n\rangle - \langle n|\widetilde{J}_t|n\rangle^2 = \sinh^2\eta\times(\Omega+n^2) = \sinh^2\eta\times(n^2-j(j-1)) > 0\,. \tag{2.6}$$

The fact that $\Delta\widetilde{J}_t \neq 0$ simply confirms that the boosted observer sees a superposition and not an proper state of length. Then, more interestingly, it depends on the chosen representation and thus could be used experimentally to choose the "right one". Let us also point out that $\Delta\widetilde{J}_t$ increases with the boost rapidity η and with the length n (so that the minimal variance is obtained for the minimal interval $n=j$), which matches our expectations.

Notice that the boosted time interval operator $\widetilde{J}_t$ will have a basis of eigenstates of the same form as that given above for the eigenstates of J_t, but with respect to a different variable, and can be decomposed as above with respect to them. The physical (or geometrical) picture is that of a cylinder centered around the t-axis: the initial slice is a circle but any oblique (or equivalently boosted) slice will be an ellipse. Now the boosted observer will describe the state on the ellipse as a superposition of states on her circle (located on the new boosted slice).

This toy model realizes explicitly although in a much simplified context the idea proposed in [11] to reconcile discrete spectra of non invariant geometric operators in flat space loop quantum gravity with the phenomenon of Lorentz contraction. The analogy with the treatment of angular momentum under rotations in ordinary quantum mechanics should be apparent.

A resulting phenomenon is that a good quantum clock, i.e. one which ticks exactly regularly, for a given observer will not be as good for any boosted observer, or more precisely, that a boosted observer with respect to a clock will not see it as indicating a definite time.

2.3 A Lorentz invariant discrete space spectrum

We now would like to also give a Lorentz invariant toy model with a fixed discrete length spectrum. Unfortunately, it can not be directly inspired from 3d lorentzian loop quantum gravity since this theory predicts a continuous spectrum for spacelike distances. However, what one can try to do is to somehow map the time interval measurement to a measurement of space distances, keeping in mind also the fact that this is after all what we always do when measuring space distances, as discussed above.

What we are going to do then is to mimic the construction described for time measurements in a euclidean setting obtained by Wick rotating the previous framework, finding an operator which has therefore the interpretation of a distance measurement operator, and then map the model back to the lorentzian domain, where the behavior of the relevant quantities under boosts is finally read out.

The main tool we are using is a map introduced by 't Hooft [13] in his study of the dynamics of $2+1$ quantum gravity, in order to study the hyperbolic geometry of spacelike slices, formulated in terms of boost (hyperbolic) angles, using usual angles. Another way to describe what we are doing is to say that, by means of the 't Hooft method, we can map the $\mathrm{SU}(1,1)$ symmetry onto a $\mathrm{SU}(2)$ symmetry, and back.

We start from a euclidean analogue of the situation described in the previous paragraph, again inspired by riemannian 3d loop quantum gravity. The spacetime length operator is given by the Casimir of the SU(2) group: $C^2 = J_x^2 + J_y^2 + J_z^2$ (the J_i's are the Lie algebra generators), which has of course always a discrete spectrum. Analogously to what we did for the time interval operator, we choose the generator J_z, to represent our toy model for non-invariant space distance operator. One can also think of an analogous mechanical picture with a spinning particle used to measure distances, but of course the picture looses much physical content as we are in a euclidean context.

Let us consider the rotation group SU(2) and choose a (infinite dimensional) representation of it, for example again that given by infinitely differentiable functions on the circle. The operator $\widehat{L}$ measuring spacelike distances will now be J_z — that we may interpret as a Wick rotated J_t operator. It is of course diagonalized easily, being represented again by a partial derivative over the angle parameter, by the same functions on the circle we used for J_t, it has discrete integer eigenvalues n and corresponding eigenvectors $\mid n\rangle$.

The riemannian analogue of Lorentz boosts will now be SU(2) rotations J_x and J_y, taken in a unitary representation of SU(2). The simplest choice is to take a spin j representation: it will contain both signs of n and will have a maximal length j. Eventually, we could take send j to ∞, or choose a infinite-dimensional representation of SU(2).

The "boosted" space distance operator is then:

$$\widehat{L}_{\text{boost}} = e^{-i\frac{\theta}{2}J_y}\widehat{L}e^{i\frac{\theta}{2}J_y} = \cos\theta\widehat{L} + \sin\theta J_x\,, \tag{2.7}$$

where θ is the rotation parameter; again this operator doesn't commute with the unbooosted one, so that any eigenstate of the former is given by a linear combination of eigenstates of the latter. Of course the "boosted" states diagonalize the "boosted" operator, just as in the lorentzian case. The relevant quantity for retrieving the usual behavior under boost is the expectation value in a given eigenstate of the space distance operator; this expectation value transforms as

$$\langle n|\widehat{L}_{\text{boost}}|n\rangle = n\cos\theta = \cos\theta\langle n|\widehat{L}|n\rangle\,. \tag{2.8}$$

Up to now of course we have just repeated the same analysis done for the time interval operator, but the link with the lorentzian picture and with the true physical significance of boost and Lorentz contraction is still hidden. In order to compare what we have got so far with the situation in the lorentzian signature we have to map from the riemannian to the lorentzian setting, to Wick rotate back. The Wick rotation map has to be of course purely algebraic, since we are working in a coordinate free framework. The map has to turn rotation operators J_x and J_y into boost operators and relate thus the rotation angle θ to the boost parameter η we we were working with in the previous paragraph.

The correct relation is:

$$\cos\theta = \frac{1}{\gamma} = \frac{1}{\cosh\eta}\,, \tag{2.9}$$

which can also be expressed as $\tan(\theta/2) = \tanh(\eta/2)$. Let us point out that this is a bijection between $\theta \in [0, \pi/2]$ and $\eta \in \mathbb{R}_+$. This is exactly the relation between angles and boost parameters introduced in [13] and it defines a Wick rotation of SU(2) into SU(1,1). Written in the spinorial representation, the map reads:

$$e^{i\frac{\eta}{2}J_y} = \cosh\frac{\eta}{2}\left(1 + i\tanh\frac{\eta}{2}J_y\right) \in \mathrm{SU}(1,1) \to e^{i\frac{\theta}{2}(iJ_y)} = \cos\frac{\theta}{2}\left(1 + i\tan\frac{\theta}{2}(iJ_y)\right) \in \mathrm{SU}(2)\,. \tag{2.10}$$

In the Minkowski spacetime, we are boosting the plane (t, x) sending the axis $t = 0$ to the line of slope $\tanh(\eta/2)$. From the point of view of a riemannian spacetime, we would like then to send the axis $t = 0$ $(z = 0)$ to the same line of slope which implies that the rotation angle is given by $\tan(\theta/2) = \tanh(\eta/2)$.

Now it is easy to check that all the relations written above in terms of θ translate into the expected relations describing the correct behavior under boosts of the space distance operator, of its eigenstates and eigenvectors, and of its expectation value.

In this way we introduced a length operator $\widehat{L}$ with discrete spectrum, and boosts operators B which send eigenstates $|n\rangle$ of $\widehat{L}$ to superpositions $\sum_n c_n|n\rangle$, but such that the expectations values of $\widehat{L}$ and $\widehat{L}_{\text{boost}} = B^{-1}\widehat{L}B$ are still related (exactly) by the classical Lorentz contraction law. The variance of the boosted state will depend on the chosen representation i.e. here the value of the maximal length j. Let us conclude on an interesting point: under such boosts, the pure state $|n\rangle$ will be mapped to a superposition of states $|m\rangle$ where m can be zero and even *negative*. This means we have a kind of tunneling effect (in the sense of a classically forbidden phenomenon which is instead allowed quantum mechanically): if one observer sees a particle A on the left of the particle B, then there is a non-vanishing probability that the boosted observer will see them superposed or even the particle A on the right of particle B. This is the quantum counterpart of the classical phenomenon of length contraction.

3. Length in deformed special relativity

Deformed special relativity (DSR) [14] was especially introduced to address the issue of constructing a relativistic theory, i.e. one in which inertial observers see equivalent physics, with the equivalence being given by a group of transformations with both a invariant speed c and a invariant length l_P. A feature of such a theory is now that the speed of light depends on the energy E of the beam and that c is only the speed of light as $E \to 0$ [14]. The theory is still Lorentz invariant, even though the action of the Lorentz

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transformations becomes non-linear, and the structure of the translations is modified. The underlying symmetry algebra was understood to be the κ-deformed Poincaré algebra [16], and spacetime becomes non-commutative. Different bases for the κ-Poincaré algebra give rise to different formulations of the theory, whose physical equivalence or difference is not clear at present. Also, and this will be a key point in the following, a DSR theory can be characterized as a theory of a single particle system with modified phase space, with a curved but maximally symmetric momentum sector, and a non-commutative flat spacetime sector. the different bases mentioned above can be understood as different choices of coordinates on the curved momentum sector. Many of the properties of such a theory, including the non-commutativity of spacetime, the curved (De Sitter) structure of the space of momenta, the fundamental Lorentz invariance of the now quantum geometric picture were first understood and described by Snyder [1].

The general interpretation of DSR theories is that they represent an effective description of a sector of a full quantum gravity theory, in which one neglects the dynamical aspects of the gravitational field, so works on a fixed quantum gravity state, but retains at least some of its quantum properties, namely the existence of a minimal invariant length, usually identified with the Planck length, and also the fact that spacetime distances are now observable properties of the gravitational field, therefore quantum operator observables, thus in general non-commuting. However, the precise way in which DSR theory are supposed to arise from full quantum gravity is not clear, also since a complete quantum gravity theory does not exist as yet. We have good candidates for it, loop quantum gravity being one of these, and some work on how the DSR effective description may come about from loop gravity was done recently [18]–[20], with the most solid results having been obtained in the 3d case [19, 20].

Let us note that DSR is not yet fully understood and that there exists different proposals about the physics of the theory. In this context, one should view our analysis as an attempt to clarify a few points of the corresponding physical theory, and more precisely about the quantum geometric structure of spacetime underlying it. We discuss first measurements of distances in the DSR framework, and the problems that an operational definition presents. Second we give a working geometric definition of distances and distance quantum operators, study their spectral properties and their behavior under boosts, investigating what happens to length contraction in this context. Finally we analyze the algebraic structure of the mathematical theory in order to understand how to localize points in a non-commutative spacetime. For this purpose, we introduce wave packet states to represent the best possible definition of points of a quantum spacetime, and study some of their properties. All this will be done in the two most common bases (or versions) of DSR theory: the Snyder basis and the kappa-Minkowski basis.

3.1 Distance measurements in VSL theories

Let us consider distance measurements in a DSR theory, using the same radar experiment as usual. DSR theories, in some choices of bases, predict a varying speed of light (VSL): the speed of light depends on the energy of the photon (it converges to c only at low energy $E \to 0$ and goes to ∞ when E reaches the Planck energy). When this does not happen,

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the usual operational definition as in special relativity applies, and the difficult issues are those resulting from the non-commutativity of spacetime coordinates, i.e. how to define spacetime distance operators and spacetime points. When this happens, instead, there are clearly additional complications, since the speed of light c is use to define the unit of length.

In more details, simply measuring the time of travel of the photon will give us an effective distance:

$$d_{\rm eff}(E) = \frac{T_{(E)}}{2} \times c\,, \tag{3.1}$$

which would be different from the “true” distance:

$$d = \frac{T_{(E)}}{2} \times c(E)\,. \tag{3.2}$$

Then assuming that, generically, the speed $c(E)$ increases with the energy E (with $c(E \to 0) = c$ and $c(E \to E_P) = \infty$), $d_{\rm eff}$ would decrease even though d should remain constant (as a property of the physical system and not of the measurement).

As result, the measurement of a single distance does not make sense. Only ratios of distances contain valuable information. Working at a given energy E, one can compare the length of two different rulers, since their ratio will not depend on the speed $c(E)$. Of course, apart from the more involved context, this statement is not valid only for DSR but also applies to ordinary special relativity.

Now, one might want to get the curve $c(E)$. There is no real way to get $c = c(E \to 0)$, and we can once again only have access to the ratio $c(E')/c(E)$. At this point, we should also remember that we have a natural uncertainty in the distance measurements:

$$\delta d_{\rm eff}(E) \geq \lambda(E)\,,$$

with $\lambda(E)$ goes to ∞ when $E \to 0$ and decreases when E grows. One does a more precise distance measurement by using a beam of light of higher energy. As a result, we naively face a problem: how to get a precise measurement of the speed of light $c(E \to 0)$? We just have to use a different reference energy E_0 such that $\lambda(E_0)$ is reasonable, and then we only compute distance or speed ratios, so that it doesn't really matter which reference energy we use.

Now, let us have a look at the Lorentz length contraction and discuss the measurement of the length of a ruler made by different observers. Now DSR theories are understood to still be Lorentz invariant. However, the representation of the Lorentz transformations is modified (and becomes non-linear). This means that, although we do not expect a modification of the physical phenomenon, we still do expect a modification of the relation between the speed v of the boosted observer and the contraction factor γ (or equivalently between the speed v and the boost rapidity or angle η). Moreover, this is assuming that the two observers agree on the energy they will use. Else they will use different speeds of light and thus obtain different (effective) distances. For example, in the special case where the two observers use the same beam of light, the energy of the light beam will not be the same in the two reference frames and that will lead to an extra correction to the Lorentz contraction law.

Thereofore, we have here two possible procedures to test the predictions of DSR. Both depend on the details of the chosen DSR theory. The first is the measurement of the curve $c(E)/c(E_0)$. And the second is the law of length contraction. Looking only at the first experiment will only calibrate the function $c(E)$ and tell us we are in a VSL theory. Only a consistency check between the results of the two experiments could be considered as a first check of the DSR framework.

We shall let aside these operational considerations from now on and tackle the issue of distance measurements in a more abstract and purely quantum geometric way.

3.2 A Lorentz invariant de Sitter lattice structure for spacetime

The first version of what we now call Deformed (or doubly) special relativity was introduced by Snyder as an example of a Lorentz invariant *and* discrete quantum spacetime [1]. The idea is to consider the spacetime coordinates as operators, with discrete spectra, acting on a curved space of momenta, where this curvature governs the operator nature and therefore the non-commutativity of spacetime. More precisely, momentum space is described as a smooth manifold with De Sitter geometry, also understood as the homogeneous space $\mathrm{SO}(4,1)/\mathrm{SO}(3,1)$, therefore with a transitive action of the $\mathrm{SO}(4,1)$ group; this symmetry group is then decomposed according to the Cartan decomposition into the Lorentz subgroup $\mathrm{SO}(3,1)$, with canonical action on the homogeneous space, and the remainder, generated by operators x_1, x_2, x_3, t giving translation on the De Sitter space, and interpreted as coordinates of a dual flat 4-dimensional spacetime, which then results in being non-commutative.[4]

If one wants a coordinate presentation of the above, the dS space of momenta is defined as embedded in a 5-dimensional Minkowski space by

$$dS = \{\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 = -\eta^2\},$$

and the spacetime coordinates are defined as:

$$x_i = ia\left(\eta_4 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_4}\right) \qquad t = x_0 = ia\left(\eta_4 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_4}\right),$$

where a is a length. a is usually taken as the Planck length l_P, but can be more simply considered as the length resolution of a particular observer [18]. The Lorentz algebra is undeformed and the generators are explicitely given by

$$J_{ij} = i\hbar\left(\eta_j \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_j}\right), \qquad K_i = i\hbar\left(\eta_0 \frac{\partial}{\partial \eta_i} + \eta_i \frac{\partial}{\partial \eta_0}\right),$$

where J_{ij} and K_i are the rotation and boost generators of the $\mathrm{SO}(3,1)$ subgroup, respectively. They act as usual on the spacetime coordinates (the action is NOT deformed), but

[4] We focus here on the 4-dimensional case, because it is the one of greatest physical interest, for reasons that should be obvious. DSR theories can of course be easily generalised to any dimension having as momentum space the higher De Sitter constructed as homogeneous space $SO(n,1)/SO(n-1,1)$. Keeping this in mind, it is easy to realize that our toy model presented in the previous section can be seen formally as a kind of (riemannian) 2d DSR.

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the *spectrum of the x_i operators is discrete* and is $\mathbb{Z}a$, while the spectrum of t is still $\mathbb{R}$. Thus *a is an universal length scale seen by all observers.* Nevertheless, as a consequence of these definitions, spacetime becomes non-commutative and the coordinate commutators are:

$$[x_i, x_j] = \left(i\frac{a^2}{\hbar}\right) J_{ij}\,, \qquad [t, x_i] = \left(i\frac{a^2}{\hbar}\right) K_i\,. \tag{3.3}$$

All operators act on the homogeneous space $\mathrm{SO}(4,1)/\,\mathrm{SO}(3,1)$, so that the Hilbert space of our DSR theory is the space of L^2 functions on $\mathrm{SO}(4,1)/\,\mathrm{SO}(3,1)$. It is then natural to introduce the conjugate momenta operators p_i, p_t as a choice of coordinates on the dS space.[5] They act multiplicatively, and have a continuous spectrum. The commutators between the x's and the p's get corrections in a^2, so that the action of the Poincaré translations is deformed. Nevertheless, when $|p| \ll \hbar/a$, then we recover the usual quantum phase space.[6]

In this context of a non-commutative spacetime with discrete coordinates, we want to discuss the behavior of measured distances and time intervals under Lorentz boosts, and the phenomenon of length contraction. Of course, in order to do this, we have to give a definition of spacetime lengths, space distances and time intervals. As we have discussed, an operational definition, already quite difficult to give in a general quantum geometric context, is made even more tricky here in a DSR context by the fact that the velocity of light is not constant. However, we can rely on the algebraic and geometric picture outlined above to give a reasonable, albeit formal, definition: the very spacetime coordinate operators are interpreted as (non-invariant) time or space distances, describing possible geometric measurements carried out by an observer living on a non-commutative spacetime, to whom this choice of coordinates is relative.

Accordingly, the spacetime length operator (square), or invariant distance (square), is given by the operator:

$$\widehat{L}^2 = -\,t^2 + x_1^2 + x_2^2 + x_3^2 \tag{3.4}$$

which is simply the difference of the two quadratic Casimirs of $\mathrm{SO}(4,1)$ and $\mathrm{SO}(3,1)$, as

[5] Mathematically, the choice of momenta is arbitrary, it will need to be physically motivated. Nevertheless, a canonical choice is:

$$p_i = \frac{\hbar}{a}\frac{\eta_i}{\eta_4}\,, \qquad p_t = \frac{\hbar}{a}\frac{\eta_0}{\eta_4}\,.$$

Then the commutation relations are

$$[x_i, p_i] = i\hbar\left(1 + \left(\frac{a}{\hbar}\right)^2 p_i^2\right), \quad [x_i, p_j] = i\hbar\left(\frac{a}{\hbar}\right)^2 p_i p_j\,, \quad [t, p_t] = i\hbar\left(1 - \left(\frac{a}{\hbar}\right)^2 p_t^2\right), \ldots .$$

Another very interesting choice, in a sense closer in spirit to the point of view advocated here, is presented and discussed in [22], and it is basically given by the usual orispherical coordinates on the De Sitter hyperboloid, following from the Cartan decomposition of the $\mathrm{SO}(4,1)$ group [23].

[6] One can easily compare the present situation with the one in a straightforward effective quantization of the flat space of ordinary special relativity, where one has a flat space of momenta on which the Poincaré group of symmetries $ISO(3,1)$ acts transitively, and the coordinates of spacetime can be identified with the generators of the translation part of this group; the resulting quantum flat spacetime is commutative, since the algebra of translations is abelian, and the space distance and time interval operators have all continuous spectrum.

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appropriate for an invariant operator defined on the coset $\mathrm{SO}(4,1)/\mathrm{SO}(3,1)$, and has the familiar form of a flat space spacetime interval. Note also that from the geometric point of view this operator is simply the d'alambertian invariant operator on de Sitter space.

Let us start by considering its spectrum. As we work in the p-polarisation and we consider the x's are (translation) operators on the hyperboloid, the wavefunctions are functions on De Sitter space, i.e. the Hilbert space of the theory is given by L^2 functions on De Sitter space. A basis of such functions is provided by the canonical basis in the vector space of simple representations of $\mathrm{SO}(4,1)$, i.e. those who have a $\mathrm{SO}(3,1)$-invariant vector (see in appendix for details about the representation theory). These representation are labeled by either an integer parameter n or by a real positive parameter ρ. More precisely, let us choose a simple representation $\mathcal{I}$ of $\mathrm{SO}(4,1)$ and note v a $\mathrm{SO}(3,1)$ invariant vector and w another arbitrary vector in the same representation. Then the function:

$$f^{(\mathcal{I})}_{v,w}(g \in \mathrm{SO}(4,1)) = \langle w | D^{\mathcal{I}}(g) | v \rangle \tag{3.5}$$

is truly a function on the coset $\mathrm{SO}(4,1)/\mathrm{SO}(3,1)$, thus on De Sitter space. Such functions generate, varying w, the space of L^2 functions over De Sitter space, as any function on $\mathrm{SO}(4,1)/\mathrm{SO}(3,1)$ can be decomposed into a linear infinite (as the representation of $\mathrm{SO}(4,1)$ in the space of functions on the above coset is infinite dimensional) combination of such basis functions by harmonic analysis (see [23]–[25]), this decomposition involving a sum over the discrete parameter n and an integral over the real parameter ρ.

Now, choosing a vector $w_{\mathcal{L}}$ living in a given $\mathrm{SO}(3,1)$ representation $\mathcal{L}$ within $\mathcal{I}$, it is easy to check that the length operator will act diagonally on $f_{v,w}$:

$$\widehat{L}^2 | f^{(\mathcal{I})}_{v,w_{\mathcal{L}}} \rangle = \left[C_{\mathrm{SO}(4,1)}(\mathcal{I}) - C_{\mathrm{SO}(3,1)}(\mathcal{L}) \right] | f^{(\mathcal{I})}_{v,w_{\mathcal{L}}} \rangle \, . \tag{3.6}$$

For a $\mathrm{SO}(4,1)$-representation $\mathcal{I} = (0,\rho)$ from the continuous series, it decomposes onto $\mathrm{SO}(3,1)$-representations of the type $(0, \tau \in \mathbb{R}_+)$, more precisely (see [26] for the full analysis),

$$R^{(0,\rho)}_{\mathrm{SO}(4,1)} = 2 \int_0^{\infty} d\tau \, R^{(0,\tau)}_{\mathrm{SO}(3,1)} \, ,$$

and the corresponding (spacetime) length eigenvalues are:

$$\widehat{L}^2 \, | f^{\rho}_{v,w_\tau} \rangle = a^2 \left[-\rho^2 - \frac{9}{4} + \tau^2 + 1 \right] | f^{\rho}_{v,w_\tau} \rangle \, . \tag{3.7}$$

A representation $\mathcal{I} = (n,0)$ from the discrete series get decomposed [26] as:

$$R^{(n,0)}_{\mathrm{SO}(4,1)} = \bigoplus_{m=0}^{n} R^{(m,0)}_{\mathrm{SO}(3,1)} + \int_0^{\infty} d\tau \, R^{(0,\tau)}_{\mathrm{SO}(3,1)} \, , \tag{3.8}$$

so that the corresponding (spacetime) length eigenvalues are[7]

$$\widehat{L}^2 \, |f^n_m\rangle \; = \; a^2 \left[n(n+3) - m(m+2) \right] \, |f^n_m\rangle \, , \tag{3.9}$$

$$\widehat{L}^2 \, |f^n_\tau\rangle \; = \; a^2 \left[n(n+3) + \tau^2 + 1 \right] \, |f^n_\tau\rangle \, . \tag{3.10}$$

Notice that due to the constraint $m \leq n$, $L^2(\mathcal{I} = n)$ is always positive, and thus the discrete simple representations correspond to time-like spacetime intervals.

Notice that, overall, there is no real discreteness of spacelike or timelike spacetime intervals. Moreover, as a same value of L^2 can be obtained using different states, one can wonder about the physical (or geometrical) interpretations of these states. Indeed maybe we only use a particular subspace of states when dealing with an experiment measuring the length of a ruler. Or more generally, it is likely that the particular physical phenomenon under consideration will dictate us which class of states we need to use (to get more physical insight, it would be interesting to compute the average values of the x spacetime coordinates). Then we will get different behaviors of the length operator. For example, if we restrict ourselves to a fixed $\mathrm{SO}(4,1)$ representation n, then we are only dealing with positive L^2 i.e. spacelike intervals, and moreover it turns out that distances smaller than $n \times a$ are quantized while distances larger than $n \times a$ remain continuous. A potential use of this choice of subspace is when one wants to fix a semi-classical length scale above which spacetime looks classical and smooth but beyond which the quantum structure of spacetime becomes relevant. The other example is when one selects the subspace defined by a fixed $\mathrm{SO}(4,1)$ representation ρ, the length spectrum is always continuous but there now exists a maximal timelike interval of length $\rho \times a$: it could be useful when studying (geometrical) properties of a (gravity) system in a finite range of proper time.

One can also look the action of other operators such as the coordinates. These, as we said, have the interpretation of non-invariant (abstract) space distance measurements, giving the possible locations of an object with respect to the observer to which the Snyder's choice of coordinates is associated. To diagonalise the operator x_μ, one uses the basis diagonalising $J_{4\mu}$ in the representation $\mathcal{I}$. As it is impossible to diagonalise the generators $J_{4\mu}$ simultaneously, one can never diagonalise the spacetime coordinates simultaneously, as it is also clear from the commutation relations among them. This of course does not prevent from defining space and time distances separately, although their operational meaning is tricky (to say the least) due to this non-commutativity. We have already mentioned this definition above and given the spectrum of the coordinate operators, defining possible results of these hypothetical measurements. However, it is clear that the x_i's, as we have mentioned above, have discrete (equispaced) spectra, while the t operator, being generator of a boost, has a continuous one, whatever irreducible representation one chooses for $SO(4,1)$ acting on the De Sitter space of momenta.

[7] This is for to the 4-dimensional case; in the general d-dimensional case the invariant spacetime length operator is given by $\widehat{L}^2 = C_{\mathrm{SO}(d,1)}(\mathcal{I}) - C_{\mathrm{SO}(d-1,1)}(\mathcal{L})$, the representations of the relevant groups are again labeled by either an integer or a real parameter, the decomposition into irreps of the subgroup is analogous and the corresponding eigenvalues are: $a^2 \left[-\rho^2 - \left(\frac{d+1}{2}\right)^2 + \tau^2 + \frac{d^2}{4} \right]$ in the case of continuous representations of $SO(d,1)$, and $a^2 \left[n(n+d-1) - m(m+d-2) \right]$ or $a^2 \left[n(n+d-1) + \tau^2 + \frac{d^2}{4} \right]$ for the discrete series.

One can also introduce the notion of a *space distance* $\widehat{l^2} = x_i x_i$, although this cannot have the meaning of a space distance at a given time, since measuring the former to a given precision requires a large indeterminacy in the latter: as $[t, x_i] \neq 0$, knowing *where* an event occurred implies not knowing *when* it did!. This space distance operator $\widehat{l}$ is a Casimir of the SO(3) subalgebra of (space) rotations and has a discrete spectrum. In order to diagonalise $\widehat{l}$, one needs to decompose the SO(4,1) representations $\mathcal{I}$ into SO(3) representations.[8] Let us underline the somehow counter-intuitive fact that $[x_i, \widehat{l^2}] \neq 0$.

Now, having set up the basic quantum geometric setting for space an time measurements, one can start investigating the behavior of these operators under Lorentz boosts, and how the usual phenomena of length contraction and time dilatation look like. It will be clear that the situation is very much analogue to the one we pictured in our toy model of time measurements, inspired by loop quantum gravity. We concentrate on space measurements and length contraction, since it is the discreteness of geometric spectra that we are most concerned with, when dealing with Lorentz invariance. The analogous analysis can be performed for the time measurements as well.

To discuss the issue of Lorentz contraction, we needs to compare operators with their boosted counterpart. To start with, let us imagine that an observer has a ruler in the direction x_1, with one end of the ruler in the same position as the observer and of course at rest with it. The states of the ruler can be decomposed on eigenstates of $\widehat{x_1}$. On the other hand, a boosted observer will decompose the ruler states on eigenstates of $x_1^{(boosted)} \equiv e^{i\eta K_1} x_1 e^{-i\eta K_1}$. Then it is straightforward to check that:

$$[x_1^{(\mathrm{boosted})}, x_1] = [\cosh\eta\, x_1 + \sinh\eta\, x_0, x_1] = \sinh\eta \left(\frac{a^2}{\hbar}\right) K_1 \neq 0\,. \tag{3.11}$$

This is the same feature as introduced in our toy model: a proper state of length for a given observer can not be a proper state for a boosted one. Notice in fact that the toy model can be seen formally as a kind of 2-dimensional version of DSR, so it is not so surprising that the behavior of geometric operators under boosts is similar. Now, telling which state will the boosted observer see given the initial state is a more involved problem with respect to the to toy model presented in the previous section, due to the more complicated action of the other coordinate operators on the eigenstates of one of them. Therefore it is not as straightforward as it was in that case to check that we recover the usual length contraction (or time dilatation) law for the mean values of the space (or time) distance operators, and a more detailed analysis would be needed. However the general picture is the same and it is again the non-commutativity shown above that allows to maintain Lorentz invariance in this quantum setting in the presence of discreteness of geometry.

[8] A simple SO(4,1) representation can be decomposed into simple SO(3,1) representations. Then a SO(3,1) representation (n, ρ) decomposes as:

$$R^{(n,\rho)}_{\mathrm{SO}(3,1)} = \bigoplus_{j=n}^{\infty} R^{j}_{\mathrm{SO}(3)}\,.$$

The first issue to solve to have a complete treatment of such problem would be to define time slices, in order to define the spaces associated to each observer and write the state of the ruler projected onto the respective slices. Of course, as time doesn't commute with the space coordinates, we will need a concept of approximate time slices. More generally, one would also like to define approximate points: a class of coherent states, representing semi-classical spacetime points, minimizing the coordinates uncertainty relations; how this can be done will be sketched later in this section.

Defining boosted time slices requires computing the action of boosts over the coordinate x_0:

$$x_0^{(\mathrm{boosted})} = \cosh\eta x_0 + \sinh\eta x_1\,, \qquad [x_0^{(\mathrm{boosted})}, x_0] \neq 0\,.$$

A ideal time slice would be defined as an eigenspace of x_0 corresponding to a given eigenvalue. Now, it is clear that we need to thicken these slices in order to be able to localize space coordinates, due to the commutation relations between them. Let us call such subspaces $T(t), t \in \mathbb{R}$. We also define the time slices $T^{(b)}(t)$ relative to the boosted observer and its time coordinate $x_0^{(\mathrm{boosted})}$. A natural requirement (and easy to satisfy) is that boosts send $T(t)$ onto $T^{(b)}(t)$: here $T^{(b)}(t) = e^{i\eta K_1}T(t)$. Then let us pick up a state $|\Psi\rangle$. The state as seen by the first observer will be the projection of $|\Psi\rangle$ onto $T(t=0)$ and we will be looking at the average of the operator x_1. The boosted observer will see the projection of $|\Psi\rangle$ onto $T^{(b)}(t=0)$ and will consider the average of $x_1^{(b)}$. To reproduce the exact setting of the Lorentz length contraction, we would need to choose a state picked around a value of x_1 and completely delocalized in x_0 i.e. representing a particle trajectory (the worldline of the end of the ruler).

To make the argument more generic, one can work with the introduced space length operator $\widehat{l}^2$ in which case it is straightforward to check that $[\widehat{l}^2_{(\mathrm{boosted})}, \widehat{l}^2] \neq 0$.

Let us now come to another subtle point concerning the quantum geometry of this non-commutative spacetime setting, related to the previous one of constructing time slices. What is a spacetime point? How can an observer localize any event, even in principle, let alone the operational specification of such a localization procedure?

Clearly, due to the commutation relations among spacetime coordinates, that we identified with space and time distance measurements with respect to a point (origin) where the observer using such a coordinate system is, it is not possible to assign exact values for all these coordinates; this means that no system can be described by a state which is simultaneously eigenstate of all of these operators, no system can be perfectly localized in space and time, although its spacetime distance from the observer can be given exactly (the state describing this system may be labeled by a given irreducible representation of the SO$(4,1)$ group). What is then the state of a system, if we have localized it as much as the framework allows us to do? This is a familiar problem in Quantum Mechanics, where classically commuting observables do not commute at the quantum level (think for simplicity of coordinate and momentum in a given direction), and it can be dealt with in the usual way: such a state must be given by a wave packet in spacetime, for a given spacetime distance, i.e. by a state living in a given representation of SO$(4,1)$, and that is peaked around some eigenvalue of the space and time coordinate operators with a given small dispersion. This would be our definition of a "spacetime point" in this non-commutative setting.

Let us now discuss the construction of these states. The relevant commutation relation is $[x_\mu, x_\nu] = i\frac{a^2}{\hbar}J_{\mu\nu}$. Focusing on a particular space coordinate x_i and the time coordinate t, the relative indeterminacy in space and time coordinates follows:

$$\delta_\Psi t\, \delta_\Psi x_i \;\geq\; -\frac{i}{2}\langle [t, x_i]\rangle_\Psi = \frac{a^2}{2\hbar}\langle K_i\rangle_\Psi\,. \tag{3.12}$$

This is the only equation limiting the values we can assign simultaneously to the coordinates x_0 and x_i in the given state $\mid \Psi\rangle$. We see that there is a fundamental indeterminacy as soon as the state is such that the boost operator K_i has a non-zero mean value in it. One could think of fixing this mean value $\langle K_i\rangle_\Psi$, say equal to k, and treat it as a fixed datum in determining the dependence of the state Ψ itself on the coordinate operators t and x_i, building it in such a way as to minimize the above uncertainty relation with a constant $\frac{a^2 k}{2\hbar}$ on the right end side. Then we could write coherent states as for the harmonic oscillator. Unfortunately, it doesn't seem that it provides a good approximation for solving the problem and we will need to take into account the dependence of $\langle K_i\rangle_\Psi$ on the state Ψ. This easier shortcut being unavailable, we should then use the tools of coherent states for Lie groups (see for example [27]) to write down coherent states for localized spacetime points, defining them in this case as those coherent states for the group $SO(4,1)$ invariant under the $SO(3,1)$ subgroup that are closest to the classical values. We give the general ideas of this construction and some examples of it in the following subsection.

3.3 Lie groups' coherent states to localize points

A general construction for all Lie groups can be found in [27]. The procedure for constructing a system of coherent states is as follows: given a Lie Group G and a representation T of it acting on the Hilbert space $\mathcal{H}$, with a fixed vector $\mid \Psi_0\rangle$ in it, the system of states $\{\mid \Psi_g\rangle = T(g) \mid \Psi_0\rangle\}$ is a system of coherent states. The point is then to identify which are the states Ψ_0 that are closest to classical, in the sense that they minimize the invariant dispersion or variance $\Delta = \Delta C = \langle \Psi_0 \mid C \mid \Psi_0\rangle - g^{ij}\langle \Psi_0 \mid X_i \mid \Psi_0\rangle\langle \Psi_0 \mid X_j \mid \Psi_0\rangle$, where $C = g^{ij}X_iX_j$ is the invariant quadratic Casimir of the group G (that, remember, we have taken to represent the invariant spacetime length in our quantum geometric setting). The general requirement is the following: call $\mathfrak{g}$ the algebra of G, $\mathfrak{g}^c$ its complexification, $\mathfrak{b}$ the isotropy subalgebra of the state $\mid \Psi_0\rangle$, i.e. the set of elements b in $\mathfrak{g}^c$ such that $T_b \mid \Psi_0\rangle = \lambda_b \mid \Psi_0\rangle$, with λ_b a complex number, and $\bar{\mathfrak{b}}$ the subalgebra of $\mathfrak{g}^c$ conjugate to $\mathfrak{b}$; then the state $\mid \Psi_0\rangle$ is closest to the classical states if it is most symmetrical, that is if $\mathfrak{b} \oplus \bar{\mathfrak{b}} = \mathfrak{g}^c$, i.e. if the isotropy subalgebra $\mathfrak{b}$ is maximal. This construction is covariant and indeed completely general, and it applies also to the case of our present interest, i.e. constructing coherent states for the group $SO(4,1)$ invariant under the subgroup $SO(3,1)$. However, to construct these states explicitly is no trivial task.

Let us start by giving the example of SU(2) coherent states. The goal is to minimize the uncertainty relations $\delta J_i.\delta J_j \geq \langle J_k\rangle$ or equivalently the invariant variance $\Delta =$

$\langle J_iJ_i\rangle - \langle J_i\rangle\langle J_i\rangle$. For SU(2), the coherent states (or semi-classical states) are the states of highest weight i.e. in the standard basis $|j, m = \pm j\rangle$ (for any choice of z-axis) and the rotated $g|j, m = j\rangle$ for all $g \in \mathrm{SU}(2)$. These same states can also be constructed out of the representation of SU(2) as harmonic oscillators.[9] The fuzzyness of the state is then quantified by:

$$\Delta = \langle J_iJ_i\rangle - \langle J_i\rangle\langle J_i\rangle = j(j+1) - j^2 = j\,. \tag{3.13}$$

Computing the different components for $|j, j\rangle$, we get $\Delta_z = \langle J_z^2\rangle - \langle J_z\rangle^2 = 0$ and $\Delta_x = \Delta_y = j/2$. And it is straightforward to check that they minimize the Heisenberg-type uncertainty relations. More precisely, these states $g|j, m = j\rangle$ can be seen as semi-classical states on the 2-sphere $\mathcal{S}_2$ of radius j. Interpreting the distance as given by $l = j \times l_P$ and the uncertainty of the measurement as given by $\delta l = \sqrt{\Delta} l_P$, we are finally lead to:

$$\delta l = \sqrt{j} l_P = l_P \sqrt{\frac{l}{l_P}}\,. \tag{3.14}$$

Hence we have recovered a similar relation of the expected shape (2.1) with $\alpha = 1/2$, which is supposed to correspond to the 3d case.

Interestingly, we recover the same relation in the lorentzian case using SU(1,1) instead of SU(2). The invariant variance is given by $\Delta = \langle J_t^2 - J_x^2 - J_y^2\rangle - \langle J_t\rangle^2 + \langle J_x\rangle^2 + \langle J_y\rangle^2$. Some semi-classical states minimizing Δ are given by the lowest weight states of the discrete series of representations. Such representations are labelled by an (half-)integer $n \geq 1$ (and a parity $\pm$), and the states are the $|n, m = n\rangle$ (diagonalizing J_t with eigenvalue $m = n$) for all choice of t-axis. These states can be interpreted as semi-classical states on the upper part of the 2-sheet hyperboloid in the 3d Minkowski space of radius n. Then

$$\Delta = n(n-1) - n^2 = -n\,,$$

which gives an uncertainty $\delta l = \sqrt{|\Delta|} l_P = \sqrt{n} l_P$ on the distance $l = n \times l_P$.

Then we should use the same techniques of Lie group coherent states to build coherent states localizing points in 3d and 4d DSR. Here the explicit identification of the coherent states closest to classical is more complicated, and the non-compactness of the isotropy

[9] SU(2) can be represented in terms of two harmonic oscillators $a, a^\dagger$ and $b, b^\dagger$:

$$J_z = \frac{1}{2}(a\dagger a - b\dagger b), \qquad J_+ = a\dagger b, \qquad J_- = J_+\dagger = ab\dagger\,.$$

The spin j is given by the Casimir operator $N = \frac{1}{2}(a\dagger a + b\dagger b)$ which commutes which the J's. Coherent states for the system of two harmonic oscillators are:

$$|z_a z_b\rangle = \sum_{n_a, n_b} \frac{z_a^{n_a}}{n_a!}\frac{z_b^{n_b}}{n_b!}|n_a n_b\rangle\,,$$

and they can projected onto a spin j representation by imposing that $N = (n_a + n_b)/2 = j$. The resulting state can be parametrized solely by $\xi = z_a/z_b \in \mathbb{C}$ and the mean values of the J's are:

$$\langle J_z\rangle = j\frac{1-|\xi|^2}{1+|\xi|^2}, \qquad \langle J_x\rangle = j\frac{2\mathrm{Re}(\xi)}{1+|\xi|^2}, \qquad \langle J_z\rangle = j\frac{2\mathrm{Im}(\xi)}{1+|\xi|^2},$$

which is simply the parametrization of the 2-sphere of radius j as the Riemann sphere.

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subgroup makes the details of the construction even more involved. The goal would be to minimize the fuzzyness $\Delta = \langle x_\mu x_\mu\rangle - \langle x_\mu\rangle\langle x_\mu\rangle = \langle\widehat{L}^2\rangle - \langle x\rangle^2$. The first step would be to compute the mean values of the x_μ on the spacetime length eigenvectors. After a careful analysis, we would then be able to extract the uncertainties δl in the measure of distances in DSR and check whether they have the expected behavior. For example, taking states $|f^n_m\rangle$, *assuming* that the expectation values of the x_μ vanishes, i.e. that they are states centered around the origin, then semi-classical states would be defined as minimizing the spacetime length: it would be states with $m = n$. Then $\delta x = \sqrt{\Delta} = l_P\sqrt{n}$ which has the same behavior as the SU(2) coherent states. At the end of the day, we expect that such a notion of semi-classical coherent states for DSR will always give $\delta l = l_P\sqrt{l/l_P}$ with $\alpha = 1/2$ and never an exponent $\alpha = 1/3, \ldots$ but this is only a conjecture. Further analysis is needed to confirm this and to construct the coherent states exactly.[10]

Let us stress the full covariance of this, albeit complicated, construction of coherent states minimizing the uncertainty relations.

3.4 About an anti de Sitter lattice structure

Snyder's idea can be also applied to the homogeneous space defined as SO(3, 2)/ SO(3, 1) i.e. the AntiDeSitter space. Once again the spacetime manifold is recovered as the tangent space to the hyperboloid. However, due to the change of signature, the operators x_i corresponding to the space coordinates become non-compact (anti-hermitean) while t is represented by a compact generator (hermitean). Therefore, *space coordinates have a continuous spectrum while time gets quantized.* Let us point out that this statement is similar to results from the spin foam approach to quantum gravity [6].

The Hilbert space of the theory will be the space of L^2 functions over the AdS space, which can be generated as previously using simple representations of SO(3, 2) (i.e. containing a vector invariant under the SO(3, 1) Lorentz subgroup). Now the spectrum of spacetime lengths will be given by the difference of the SO(3, 2) Casimir and the SO(3, 1) Casimir. Nevertheless, as the signature changed, the sign of the SO(3, 1) Casimir is also changed.

Simple representations of SO(3, 2) are once again of two types, either labeled by an integer $n \in \mathbb{N}$ or by a real parameter $\rho \in \mathbb{R}_+$ (and a parity $\epsilon = \pm$). But their decomposition into SO(3, 1) representations is different than in the de Sitter case [26]:

$$R^n = \int_0^\infty d\tau\, R^\tau\,, \tag{3.15}$$

$$R^{\rho,\pm} = 2\int_0^\infty d\tau\, R^\tau \oplus \bigoplus_{m=0}^{\infty} R^m\,, \tag{3.16}$$

[10] For this purpose, we think that the representation of the algebra $so(5)$ in terms of harmonic oscillators should be useful to write coherent states, the generators being expressed using two sets of creation-annihilation operators $a_{1,2}, a^\dagger_{1,2}$:

$$X_{ij} = a_i a_j\,, \qquad X^{ij} = a^\dagger_i a^\dagger_j\,, \qquad X^k_l = \frac{1}{2}\left(a^\dagger_k a_l + a_l a^\dagger_k\right).$$

so that the spacetime length eigenvalues are:[11]

$$\frac{L^2}{a^2} = -\left[n(n+3)+\tau^2+1\right] \qquad \text{or} \qquad \rho^2+\frac{9}{4}-\tau^2-1 \qquad \text{or} \qquad \rho^2+\frac{9}{4}+m(m+2)\,. \quad (3.17)$$

The same discussion as for the de Sitter case applies here. Indeed we notice that, overall, there is no real discreteness of spacelike or timelike spacetime intervals. More precisely, as a same value of L^2 can be obtained using different states, one can wonder about the physical (or geometrical) interpretations of these states, and we expect than different physical phenomenon will be analyzed using different subspaces of the space of states. A interesting case is when restricting to a fixed $\mathrm{SO}(3,2)$ representation n: as $L^2 < 0$, we are only dealing with timelike intervals, and we have a minimal time interval of length n and then a continuum of eigenvalues.

3.5 The κ-Minkowski spacetime

From the above discussion on the Snyder's picture of a non-commutative spacetime, it should be clear that the fundamental physical and mathematical inputs are simply the fact that space of momenta becomes curved with a De Sitter geometry, the fact that, since this space is best described as a coset $\mathrm{SO}(4,1)/\,\mathrm{SO}(3,1)$, there exist a global symmetry algebra (and group) acting transitively on it, and the fact that we can identify the spacetime coordinates, therefore the configuration space of the theory, as a suitably defined "translation part" of this symmetry algebra. However, in the definition of the Snyder's picture, a choice was made in this last step, which was natural from the group theoretic point of view but rather arbitrary from the physical side: a choice of decomposition of the De Sitter algebra into a Lorentz subalgebra and a "translation part"; the choice used was the usual Cartan decomposition, but this is not the only possible. In other words, there seems to be an ambiguity with respect to what one calls *physical coordinates*. Indeed one could choose another basis for the x's and assume these are the true spacetime coordinates, and there is nothing against this from the mathematical point of view.

Now, is this ambiguity simply a formal one, an ambiguity in the description of the same physics, or is it more than this? Are different choices of coordinates equivalent from the physical point of view, or is there a good argument for discarding some of them, thus selecting what is *the* physical picture of non-commutative spacetime of DSR? If they are all physically acceptable, what is their respective meaning?

What seems to us a reasonable point of view on these issues is that all the different possible choices of coordinates are physically equivalent and to be allowed, but that they represent different pictures of the same non-commutative spacetime as seen by different observers; the point of view is then that of a *relative non-commutative geometry.*

Let us look more closely to the concept of *relativity.* Special relativity broke the classical notion of simultaneity: now two observers going at different speeds would give different accounts of events. By deforming the Poincaré group, DSR gives a special role to the origin

[11] Again, this is for the 4-dimensional case; in the d-dimensional one, the analogous decomposition gives the eigenvalues: $-a^2[n(n+d-1)+\tau^2+(\frac{d-2}{2})^2]$ or $a^2[\rho^2+(\frac{d-1}{2})^2-\tau^2-(\frac{d-2}{2})^2]$ or $a^2[\rho^2+(\frac{d-1}{2})^2+m(m+d-2)]$, depending on the representation of $\mathrm{SO}(3,2)$ we are decomposing.

of the coordinate system used to describe the events i.e. to the position of the observer (and not only its speed). Furthermore, by introducing a quantum spacetime with non-commuting coordinates, defining an observer requires providing the set of measurements she could do, i.e. providing a particular basis of the algebra. Now, the issue of choosing a basis becomes: when one does an experiment measuring a coordinate, or any other (geometric) observable, which operator are we actually measuring?

Let us study another special choice of coordinates called the κ-Minkowski basis. It is defined, with respect to the Snyder's basis, as

$$X_0 = x_0 = t \qquad X_i = x_i + \frac{a}{\hbar} K_i \,. \tag{3.18}$$

It is important to stress that this choice picks up a preferred time direction, as compared to the Snyder's basis, in fact it corresponds to a shift in the spatial coordinates but leaves intact the time coordinate, and it is, as a consequence, "less covariant". We feel that this "less covariance" affects directly all the geometric properties we are about to discuss (and the coherent states constuction outlined below) in that they result in being tied to a given (class of) observer(s).

The reason why this choice of coordinates is special is most easily understood by looking at the new commutators between spacetime coordinates. These satisfy the following commutation relations:

$$[X_0, X_i] = -iaX_i \qquad [X_i, X_j] = 0 \,, \tag{3.19}$$

so that now the coordinates satisfy Lie algebra commutation relations forming thus a subalgebra of the SO$(4,1)$ algebra. This subalgebra is interpreted as a non-commutative flat spacetime, called the κ-Minkowski space. The deformation parameter is defined as $\kappa = 1/a$. When the length scale a is set to the Planck length, κ thus becomes the Planck mass (or energy).

In this picture the space coordinates are now commutative, so that the quantum nature of the space operators is somewhat hidden, however there is an intrinsic uncertainty when measuring both the time coordinate and any of the space coordinates of an object, as the corresponding operators do not commute. This uncertainty is governed by the length scale a and actually grows when getting further from the origin. In the following, we will set $a = l_P$.

Concerning the other commutation relations, the Lorentz sector of the SO$(4,1)$ algebra is not modified, as the only change with respect to the Snyder basis is in the spacetime coordinates, and we still have the same commutation relations between the J's and the K's. Nevertheless the action of the boosts on the spacetime coordinates is now modified:

$$[K_i, X_0] = i\hbar \left(X_i - \frac{a}{\hbar} K_i \right) \qquad [K_i, X_j] = i\hbar \left(\delta_{ij} X_0 - \frac{a}{\hbar} \epsilon_{ijk} J_k \right) .$$

Of course also the relations between spacetime coordinate operators and momentum operators (with a suitable choice of coordinates on De Sitter space) is affected by the new definition of the former, the new commutators being straightforwardly computable.

Interestingly, the spectrum of the X's operators (both time and space) is now continuous! Then let us insist on the fact that the κ-Minkowski basis actually depends on the choice of a time direction which allows the split between space and time. A possible interpretation is that the κ-Minkowski coordinates are relative to an observer, when it describes physical phenomena as happening in the usual classical continuous space (not spacetime) with the time direction being defined along the worldline of the observer. In this context, it appears that if one wants to keep the notion of a classical space while having a universal length scale a existing for every observer, one needs to deform the action of the Lorentz boosts and makes time fuzzy.

The spacetime invariant length is defined again by the same operator as before, but now its expression is modified and reads:

$$L^2 = X_0^2 - \left(X_i - \frac{a}{\hbar}K_i\right)\left(X_i - \frac{a}{\hbar}K_i\right). \tag{3.20}$$

Now appears the difficulty of defining a natural notion of *space distance* out of the space coordinate operators. Naturally, if one assumes that the X_i are the space coordinates, one would introduce the following space length operator:

$$\widehat{l}_\kappa^2 \equiv X_i X_i = \left(x_i + \frac{a}{\hbar}K_i\right)\left(x_i + \frac{a}{\hbar}K_i\right). \tag{3.21}$$

Let us stress that $\widehat{l}_\kappa^2 \neq \widehat{l}^2$. Still $\widehat{l}_\kappa^2$ is perfectly invariant under the SO(3) subgroup of space rotations and thus qualifies as a notion of space distance. Nevertheless, this choice is not justified by the expression for the invariant quadratic spacetime length and thus is left with an unclear geometric interpretation. This results from the unclear geometric interpretation of the new space coordinates X_i themselves (while the interpretation of the Snyder coordinates as the non-commutative analogue of cartesian coordinates was justified by the very expression for the spacetime length operator). Notice that $\widehat{l}_\kappa^2$ has a continuous spectrum contrary to $\widehat{l}^2$.

We then want to study the behavior of the newly defined space and time distance operators under the action of the Lorentz subgroup of symmetries, and in particular of the boosts, and the resulting phenomenon of length contraction.

Nothing changes of course for the time distance operator, whose definition remains the same, on the other hand it appears that there is a correction term in the action of boosts on X_1:

$$\begin{aligned} X_1^{(\text{boosted})} &= e^{i\eta K_1}\left(x_1 + \frac{a}{\hbar}K_1\right)e^{-i\eta K_1} = x_1^{(\text{boosted})} + \frac{a}{\hbar}K_1 = \cosh\eta x_1 + \sinh\eta x_0 + \frac{a}{\hbar}K_1 \\ \Rightarrow X_1^{(\text{boosted})} &= \cosh\eta X_1 + \sinh\eta X_0 - \frac{a}{\hbar}(\cosh\eta - 1)K_1 . \end{aligned} \tag{3.22}$$

It is easy to check that here too X_1 and $X_1^{(boosted)}$ do not commute:

$$[X_1^{(\text{boosted})}, X_1] = -ia\left(\sinh\eta X_1 + (\cosh\eta - 1)X_0\right) \neq 0 .$$

Therefore again we have that if an observer sees a system at a definite distance, i.e. in a given eigenstate of the X_1 operator, a boosted observer will instead see a superposition of such states and will not assign a definite distance to the same system, and again this is the key point that makes a Lorentz invariant discreteness of the quantum geometry possible. The same difficulties we pointed out in the Snyder's case, in testing whether the usual length contraction rule is recovered for the mean values, are to be found also in this k-Minkowski basis, and a more complete treatment is left for future work.

We can again study what a point is in this new picture of non-commutative spacetime, again using the notion of wave packets to define it, and restricting the analysis to a 2-dimensional slice of configuration space, to simplify it. Now, as space coordinates do commute among themselves, the extension to the full 4-dimensional picture is even more straightforward. A quantum point, or, better, the state of an ideal system defining a given location, will then be identified with a wave packet state for which the spacetime coordinates would be as peaked as possible around a given value of the spacetime coordinates (X_0, X_i).

The commutation relations (3.19) lead to a Heisenberg uncertainty relation of the type:

$$\delta X_0 . \delta X_i \geq \frac{l_P}{2} X_i \,, \tag{3.23}$$

so that the uncertainty on the variables X_0, X_i increases with X_i, i.e. the further we are from the origin, or in other words the further the object to be localized is from the observer, the harder it is to localize it.[12] Furthermore, we expect that if an object is really well localized in time, say up to a Planck scale resolution, i.e. $\delta X_0 = l_P$, then it would be completely delocalized in space, i.e. with a delocalization equal to its distance from the observer.

There is an easy way of constructing such wave packet states; let us do a (non-linear) change of basis and introduce variables y_i such that $X_i = a \exp(y_i/a)$; then we get the simple canonical commutation relation:

$$[X_0, y_i] = [x_0, y_i] = -ia^2 = -il_P^2 \sim -i\hbar \,, \tag{3.24}$$

which correspond to the classical bracket $\{x_0, y_i\} = 1$. We would like to introduce localized states which minimize the uncertainty on x_0 and the coordinates y_i. To this purpose, we can use the usual wave packet states as built for a harmonic oscillator on the variables x_0, y_i in order to minimize the uncertainties $\delta x_0 \sim \delta y_i \sim \sqrt{\hbar} = l_P$.

Such states have the simple gaussian form:

$$f_{(x_0^0, \vec{y}^0)}(y_i) \propto e^{i\frac{x_0^0 y_1}{a^2}} e^{i\frac{x_0^0 y_2}{a^2}} e^{i\frac{x_0^0 y_3}{a^2}} e^{-\frac{|\vec{y}-\vec{y^0}|^2}{a^2}} \,.$$

In the X_i coordinates, these states have a more complicated form.

However, these are not good localized states with respect to the κ-Minkowski coordinates, as can be seen by simply looking again at the uncertainty relations: the uncertainty on the X_i's is:

$$\delta X \sim e^{\frac{y}{l_P}} \delta y \sim \frac{X}{l_P} \delta y \sim X \,. \tag{3.25}$$

Indeed we see that such states are completely delocalized in space.

[12] Let us point out that this is consistent with the viewpoint of the κ-Minkowski coordinates being relative to a particular observer standing at the origin: the further away an event is, the harder it is to determine its precise time of occurrence.

However, a simple modification of the above construction solves the problem: we introduce an additional length scale l for semi-classical physics, such that $l >> l_P$ but still much smaller that the resolution of any achievable direct measurement of localization (for example, the best resolution in time ever been achieved, and very recently, is of 10^{-16} seconds, hundreds of attoseconds, while the Planck time is of the order of 10^{-40} seconds, so there is plenty of room in between!). Now, let us build coherent states for the conjugated pair of canonical variables $(x_0\alpha, y_i/\alpha)$ with $\alpha = l/l_P$. Then the magnitude of the uncertainties are:

$$\delta x_0 = l \qquad \text{and} \qquad \delta y_i = \frac{1}{\alpha} \Rightarrow \delta X_i = \frac{1}{\alpha} X_i = \frac{l_P}{l} X_i\,. \tag{3.26}$$

Choosing $\alpha \ll 1$ such that $l_P \ll l \ll \mathcal{L}$, where $\mathcal{L}$ corresponds to our length scale (the scale of resolution of our best measurement devices) where all looks smooth and classical, we can consider such the as semi-classical states where spacetime points are well localized, or simple *localized states*.[13]

A complete analysis of the transformation properties under boosts of these wave packets is left then for future work. The issue is again how localized states with respect to the boosted coordinate operators differ from the one constructed with respect to the un-boosted ones. Let us finish by pointing out that, interpreting the κ-Minkowski basis as depending on an observer and its time direction, then in order to build coherent states for a boosted observer we would need to change basis by changing the x_0 direction in the algebra and then re-do the whole work to construct new coherent states which would be completely different than the ones constructed for the original observer. From this point of view, it is clear that the description of events by a boosted observer will be different from the description provided by the original observer. This again highlights the lack of covariance of this basis as compared to the Snyder's basis, but simply because it is attached to a particular observer.

4. Comparison of results and hints for loop quantum gravity

Here we should add a few comments on the results obtained, and a comparison between what we got in DSR case and the toy model inspired by LQG, stressing similarities in the overall picture, in spite of differences in the details.

Let us comment briefly on the results obtained in the toy loop-inspired model and in the DSR framework. The basic point that is in common to both cases is a fundamental Lorentz invariance of the setting. At the same time in both cases it is possible to give a simple definition of geometric distance operators giving rise to a discrete picture of quantum geometry. The apparent contradiction between a continuous group of symmetries and this discreteness is solved thanks to the unitary representation of this group on the Hilbert space of states of the theory and to the non-commutativity between geometric operators and their

[13]Let us point out that such a setting is one of the current effective framework for extracting semi-classical information out of a loop quantum gravity state [28]. Actually, the introduction of an additional quantity characterizing the scale of observations at which the effective description applies is a general feature of any approximation scheme.

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Lorentz-transformed counterparts. Again, this is true, and realized similarly, both in the loop-inspired toy model and in DSR. In the first framework, moreover, due to its extreme simplicity, it is possible to check explicitly that one recovers the usual transformation laws for space and time distances, i.e. the usual Lorentz contraction or dilatation, for the mean values of the relevant operators. In the DSR case the more involved details of the framework prevents from doing so. However, also in this case we would expect the usual laws to be recovered for mean values. The reason is that the DSR framework is usually interpreted as an effective description of the flat limit of Quantum Gravity; in this limit, i.e. if the quantum state of geometry under consideration is that describing flat space, we should have that the Lorentz symmetry extends from a local symmetry of the theory to a global isometry of the geometry, in other words to a global symmetry of the state, either exactly or in a macroscopic limit. Standing the quantum nature of the problem and of the setting, and thus the results found above about the behavior under boosts of the geometric operators, we expect this to be manifest exactly in the classical-like behavior of the mean values of these geometric operators. One can also turn this logic around and use the properties of geometric operators in the given state to gain information about the state itself. In other words, the presence of such a global symmetry, made clear by the classical-looking transformation properties of the expectation values of distance operators, can be used to characterize the state under consideration as that corresponding to flat space (of course also a translation symmetry has to be present in addition to the Lorentz one). This is a hint of a new possible approach to defining a 'Minkowski vacuum state' in loop quantum gravity (LQG), inspired by the analysis presented in this paper. Let us make this more explicit.

In [11], the authors defined Lorentz transformation in LQG and studied the properties of the Lorentz boosted area operator. Here we propose the idea of applying the same framework to the volume operator. Lorentz boosts are generated by the hamiltonian constraint $\mathcal{H}$ with a special choice of lapse function N depending on the boost rapidity η. The (Poisson) algebra of commutation between the volume (operator) $\mathcal{V}$ and the hamiltonian $\mathcal{H}$ is particularly interesting. Roughly it goes as:

$$\begin{aligned} \mathcal{H} &= \{S_{CS}, \mathcal{V}\}\,, \\ \mathcal{C} &= \{\mathcal{H}, \mathcal{V}\}\,, \\ \frac{2}{3}\mathcal{V} &= \{\mathcal{C}, \mathcal{V}\}\,, \end{aligned} \tag{4.1}$$

where S_{CS} is the Chern-Simons action and $\mathcal{C}$ the operator implementing changes of Immirzi parameter in LQG and thus generating scale transformation on the kinematical geometrical operators (like the volume). Now, under a small boost, the volume will get shifted to:

$$\mathcal{V}_\eta = \mathcal{V} + \{N_\eta \mathcal{H}, \mathcal{V}\} = \mathcal{V} + N_\eta \mathcal{C}\,. \tag{4.2}$$

It is obvious that $\mathcal{V}_\eta$ and $\mathcal{V}$ do not commute. Moreover their commutators is proportional to the volume itself:

$$\{\mathcal{V}_\eta, \mathcal{V}\} \propto \mathcal{V}\,, \tag{4.3}$$

which implies that the volume and boosted volume become more and more "different" when the volume grows i.e. their eigenstates are different. Now requiring the contraction law of the volume under boosts in the flat Minkowski background, $V_\eta = V/\cosh\eta$, a (localized) Minkowski state $|\psi\rangle$ should satisfy the following criteria:

$$\langle\psi|\mathcal{V}_\eta|\psi\rangle \approx \frac{\langle\psi|\mathcal{V}|\psi\rangle}{\cosh\eta} \Rightarrow \langle\psi|N_\eta\mathcal{C}|\psi\rangle \approx \frac{-\eta^2}{2}\langle\psi|\mathcal{V}|\psi\rangle\,, \tag{4.4}$$

for (macroscopic) space regions, when $\langle\psi|\mathcal{V}|\psi\rangle$ is large and η small. Further analysis of the operator $\mathcal{C}$ is required to check how meaningful this proposed criterium is.

5. Conclusion

The main goal of this paper was to show in which sense the discreteness of a quantum space-time geometry originating from Quantum Gravity may still be compatible with Lorentz invariance; this is somewhat counterintuitive, as almost everything when it comes to Quantum Gravity, but we have shown how this may be possible and easily realized if one takes seriously the quantum nature of geometric measurements in this context.

In order to achieve this result, we studied a simplified model of a quantum flat geometry, directly inspired by 3-dimensional loop quantum gravity, where it was possible to define what time and space distance measurements were and to carry through all the relevant (easy) calculations. We think this simple model can be of inspiration also for work in the complete loop quantum gravity setting.

We found that the compatibility of quantum discreteness of geometric spectra and continuous Lorentz invariance is possible due to the unitary action of the Lorentz boost operators on quantum states and distance operators, and a non-commutativity of these and their boosted counterparts. This results in the fact that the state of a localized system for a given observer turns into a de-localized one for another observer boosted with respect to the first. Our result then confirms, in this simplified context, but in full detail, the argument for resolving the apparent contrast between discrete quantum geometry and Lorentz invariance presented in [11].

We think it is also of interest the fact that a simple procedure for mapping SU(2) (riemannian) quantum geometry to the SU(1,1) (lorentzian) one, introduced by 't Hooft in [13] turned out to be useful in our analysis; this may hint to a closer than expected similarity between the loop quantum gravity/spin foam approach and his polygon-based discrete framework.

In the second part of the paper, we turned our attention to Doubly (or deformed) special relativity theories, considered as effective models of full quantum gravity. In this models Lorentz invariance is present but somehow hidden, due to a modified action of the Lorentz group, and most important the picture of quantum geometry that lies behind them is unclear. We tried to analyze this important aspect using similar tools as those which are customary in loop quantum gravity and spin foam models, basing our work on the geometric rather than on the quantum algebraic structure of DSR, in both the Snyder and the κ-Minkowski bases.

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We gave a simple geometric definition of time interval and space distance measurement operators in DSR, and analyzed the spectral properties of these and of other geometric operators; this needed a first study of the Hilbert space structure of DSR models. We then analyzed the behavior under boosts of these operators and of their spectra, showing a nice parallelism with the structure and properties of the loop inspired toy model discussed previously. Again it is the quantum nature of geometric operators that makes the co-existence of discrete quantum geometry and continuous Lorentz invariance possible. We also discussed wave packet states, as a sensible definition of quantum spacetime points in a non-commutative context.

In this way, we think we achieved a better, although far from complete, understanding of the picture of quantum geometry resulting from DSR theories, and made the comparison between the full loop quantum gravity/spin foam theory and its DSR effective description a bit easier. Also, we think the analysis presented may give useful hints on the problem of defining and characterizing a quantum state corresponding to flat space in a full theory of Quantum gravity, e.g. loop quantum gravity.

We are well aware that this can be at best a very first step towards a complete understanding of the issues dealt with in this paper, but we feel it was a *necessary* first step, and a valuable one.

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A. Length contraction in special relativity

Let us consider a ruler of length l in its referential at rest. Let us consider an observer moving at a speed v (in the direction defined by the axis of the ruler). Then the two referentials are related by a Lorentz transformation Λ and the length l' seen by the moving observer is defined as:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 0 \\ l' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ l \end{pmatrix}, \tag{A.1}$$

with $\gamma = 1/\sqrt{1+v^2} = \cosh\eta$, η being the boost rapidity. Then solving this equation, one gets the usual result: $l' = l/\gamma$.

Let us keep in mind that we are dealing with the distance between two spacetime points, but between the two worldlines of the ends of the ruler. Each observer is going to give a different values of such a distance. The Lorentz invariant notion is the concept of the *maximal* distance between the two lines, which is indeed given by the rest ruler length L (the minimal positive distance is of course 0, since there always exist a null ray going from one worldline to the other).

Let us compare this result with the true length measurement done through the photon time-of-flight experiment. For an observer $\mathcal{O}_1$ moving towards the other end of the ruler, the photon flight will be shorter than for the observer at rest $\mathcal{O}$. It is easy to check that:

$$T_1 = T\frac{1}{\gamma(1+v)} = Te^{-\eta} < T\,, \qquad \eta \geq 0\,.$$

One should note that the contraction factor is not $\gamma = \cosh\eta$ anymore. Now if an observer $\mathcal{O}_2$ moves away from the ruler, then the photon flight will be longer than for $\mathcal{O}$:

$$T_2 = Te^{+\eta} > T\,, \qquad \eta \geq 0\,.$$

B. Some facts about Snyder's DSR

To complete our presentation of DSR, let us add a few facts. In the Snyder basis, as the spectrum of some coordinate operators is discrete, one can not represent the momentum operators p_μ as derivation operators $\partial/\partial x_\mu$. On the other hand, one can use the p-polarisation and still represent the x_μ operators as derivations with respect to the momenta [1]:

$$\widehat{x_i} = i\hbar\left(\frac{\partial}{\partial p_i} + \left(\frac{a}{\hbar}\right)^2 p_i\left(p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} + p_3\frac{\partial}{\partial p_3} + p_t\frac{\partial}{\partial p_t}\right)\right), \tag{B.1}$$

$$\widehat{t} = i\hbar\left(\frac{\partial}{\partial p_t} - \left(\frac{a}{\hbar}\right)^2 p_t\left(p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} + p_3\frac{\partial}{\partial p_3} + p_t\frac{\partial}{\partial p_t}\right)\right). \tag{B.2}$$

These operators are then hermitean with respect to the deformed measure:

$$d^4\tau = \frac{\hbar\, dp_1\, dp_2\, dp_3\, dp_t}{ac\left(p_1^2 + p_2^2 + p_3^2 - p_t^2 + \left(\frac{\hbar}{a}\right)^2\right)}, \tag{B.3}$$

which is to be used in quantum mechanics (or field theory) computations. We see that it contains a singularity, or equivalently a resonance, at the Planck mass $m^2 = (\hbar/a)^2$, which is a signature of some quantum gravity effects.

Let us underline that the previous expression were given in [1] for the DeSitter case. In the DSR theory defined as an AntiDeSitter lattice, the correction would change sign and we would have a $-(\hbar/a)^2$ term in the denominator of the measure: upon restriction to purely timelike momenta, there is no divergence in the measure anymore at $p = 0$.

C. About representations of SO(2, 1), SO(3, 1) and SO(4, 1)

The Lorentz groups in 3d, 4d and 5d are respectively SO(2, 1), SO(3, 1) and SO(4, 1). We need to study their representation theory inorder to extract the properties of the length operator of doubly special relativity.Let us choose as signature $(-++\ldots)$ so that spacelike curves have a positive squared length.

SO(2, 1) has 3 generators $J_{01,02,12}$ (two boosts and one rotation). It has one Casimir operator:

$$C(\mathrm{SO}(2,1)) = J_{12}^2 - J_{01}^2 - J_{02}^2\,.$$

Its (principal) unitary representations are of two kinds:

- the continuous series labeled by a real number $s > 0$. For them, $C(\mathrm{SO}(2,1)) = -(s^2 + 1/4)$.
- the discrete series labeled by an integer $j > 1$ and a sign ϵ. Then $C(\mathrm{SO}(2,1)) = j(j-1)$.

SO(3,1) has 6 generators J_{IJ} $(I,J = 0,\dots,3)$. Its has two Casimir operators: $C(\mathrm{SO}(3,1)) = J_{IJ}^2 - J_{0I}^2$ and $\widetilde{C}(\mathrm{SO}(3,1)) = \epsilon^{IJKL}J_{IJ}J_{KL}$. Its (principal) unitary representations are labeled by a couple $(n \in \mathbb{N}, \rho \in \mathbb{R}^+)$ and their Casimir are: $C(\mathrm{SO}(3,1)) = n^2 - \rho^2 - 1$ and $\widetilde{C}(\mathrm{SO}(3,1)) = 2n\rho$. We call *simple representations* these ones that have a vector invariant under the SO(2,1) subgroup (leaving the third direction x_3 invariant). Obviously, the second Casimir then vanishes, $n\rho = 0$, and we have two series of representations: $(n,0)$ and $(0,\rho)$. Let us also point out that a representation (n,ρ) can be decomposesd into SO(3) representations and that the relevant spins are $j \geq n$.

SO(4,1) has 10 generators J_{IJ} $(I,J = 0,\dots,4)$. It also has two Casimir operators: $C(\mathrm{SO}(4,1)) = J_{IJ}^2 - J_{0I}^2$ and $\widetilde{C}(\mathrm{SO}(4,1)) = -W_0^2 + W_I W_I$ with $W_I = \epsilon^{IABCD}J_{AB}J_{CD}$. Let us notice that $W_4 = C(\mathrm{SO}(3,1))$. Simple representations are now defined as having a SO(3,1) invariant vector. It is straightforward to check that this is equivalent to $\widetilde{C}(\mathrm{SO}(4,1)) = W^2 = 0$, so that we are also left with two series of representations $(N,0)$ and $(0,R)$.

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